Pricing Liquidity Risk with Heterogeneous Investment Horizons*

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Abstract

Liquidity risk should only matter to investors insofar as they have to trade. The investment horizon should therefore be taken into account in the analysis of liquidity risk. We study both the theoretical and empirical asset pricing implications of allowing for heterogeneous holding periods in a setting where investors face stochastic illiquidity costs. More specifically, we extend the liquidity-adjusted CAPM of Acharya and Pedersen (2005) to a multi-period setting by introducing a new class of investors with longer investment horizons. We study two baseline cases: one setting with full integration, where all investors hold all assets. A second setting features market segmentation, where short-term investors only invest in assets with low transaction costs. Our theoretical framework offers many insights to characterize the sources of the liquidity premium.

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1 Introduction

The investment horizon is one of the most important features that determine asset allocation strategies. With time-varying investment opportunities, the pioneering work of Merton (1971) theorizes that the portfolio choices of multi-period investors can differ from that of a single-period investor because of hedging demands whose magnitude depends on the investment horizon of the investor. Recent asset pricing research has built on this insight to demonstrate that multi-period decisions differ substantially from single-period decisions in different model specifications (e.g., Campbell and Viceira, 1996; Balduzzi and Lynch, 1997; Brandt, 1999).

Different portfolio rules for different horizons imply different trading schemes. In particular, liquidity plays a different role for investors with different horizons insofar as trading costs only matter when trading actually takes place. The investment horizon becomes then a key element in the analysis of liquidity.

We explicitly take this standpoint and model a continuum of investors with heterogeneous investment horizons, with stochastic illiquidity costs. Specifically, our model features investors with short and long investment horizons. These latter investors are less concerned about trading costs because they do not trade every period. Furthermore, the longer investment horizon allows to earn larger risk premia that can potentially cover the higher trading costs of the more illiquid assets.

Previous theories of liquidity and asset prices have largely ignored investors' horizon, with the exception of the seminal paper of Amihud and Mendelson (1986), who study the existence of *clienteles* that have different liquidity preferences in a setting where transaction costs are constant. However, there is large empirical evidence that liquidity is time-varying. The most influential asset pricing model with liquidity risk, Acharya and Pedersen (2005), features stochastic transactions costs, but includes only one type of investors trading every period. Our paper bridges these two seminal papers, because our model entails different clienteles, as in Amihud and Mendelson (1986), with stochastic illiquidity, as in Acharya and Pedersen (2005).

This theoretical setup delivers a number of novel and interesting predictions. In an equilibrium where all investors trade all assets (integration), the existence of clienteles with longer investment horizons reduces the importance of liquidity risk relative to the standard CAPM market risk compared to a setting where all investors trade every period. More specifically, the relative importance of the two risk premiums depends crucially on the risk aversion of long-term versus short-term investors. For example, if long-term investors are more risk-averse, liquidity risk becomes more important because short-term investors, who care more about liquidity, hold a relatively larger fraction of the asset supply in equilibrium.

Another important prediction of this setup is that, given that some investors do not trade every period, the effect of expected liquidity is also smaller and it varies in the cross-section of stocks according to the covariance between returns and illiquidity costs. In an equilibrium where short-term investors do not invest in some more illiquid assets (partial segmentation), our model shows that expected stock returns contain again a larger proportion of market risk premiums relative to liquidity risk premium, plus an additional component reflecting the extent of supply for the segmented asset. The effect of expected liquidity on returns is naturally larger for the assets that are traded by all investors. In this setup, an increasing and concave relationship between expected returns and trading costs arises naturally, since excessive trading costs exclude the clientele that is more sensitive to liquidity costs.

These theoretical predictions are borne out in the empirical estimation. Specifically, we explore the cross-sectional predictions of the model using U.S. stocks over the period 1962 to 2004. We use the illiquidity measure of Amihud (2002) to proxy for liquidity costs. We find that our heterogeneous-horizon asset pricing model fares better than a standard or a liquidity-adjusted CAPM, both in terms of R-squared for cross-sectional returns and p-values in specification tests.

Our empirical results contribute to a rich literature that has shown the asset pricing implications of liquidity and liquidity risk. Amihud (2002) finds that stock returns are increasing in the level of illiquidity both in the cross-section (consistent with Amihud and Mendelson, 1986) and in the time-series. Pástor and Stambaugh (2003) show that the sensitivity of stock returns to aggregate liquidity is priced. Acharya and Pedersen (2005) integrate these effects into a liquidity-adjusted CAPM that performs better empirically than the standard CAPM. Acharya and Pedersen (2005) demonstrate that liquidity matters for asset pricing in the sense that a liquidity-adjusted CAPM performs better than the standard CAPM. The liquidity-adjusted CAPM is such that, in addition to the standard CAPM effects, the return on a security increases with the level of illiquidity and is influenced by three different sources of liquidity risks. These liquidity risks may be summarized as follows: investors require compensation for holding a security that on average is illiquid when the market is illiquid; investors are willing to accept a lower return on a security that provides a high return when the market is illiquid; and willing to accept a lower return on a security that is liquid when the market return is low.

Furthermore, our paper is also related to research showing the relations between liquidity and investors' holding periods. For example, Chalmers and Kadlec (1998) find evidence that it is not the spread, but the amortized spread that is more relevant as a measure of transaction costs, as it takes into account the length of investors' holding periods. Cella, Giannetti, Ellul (2011) demonstrate that investors' short horizons amplify the effects of market-wide negative shocks.

The remainder of the paper is organized as follows. Section 2 illustrates our multi-period liquidity CAPM in the most intuitive setting with two investment horizons (one-period and two-periods) and two assets. Section 3 generalizes the model to arbitrarily many investment horizons and assets. We describe our estimation methodology in Section 4. Section 5 illustrates the data and Section 6 presents our empirical findings. We conclude with a summary of our findings in Section 7.

2 A Two-Period Two-Assets Liquidity-CAPM

In this section we present a simple version of our asset pricing model, with two investor types and two assets. Asset i pays a dividend D_i and selling the asset costs C_i. The first investor type has a one-period horizon and mean-variance preferences with risk-aversion

A_1. At time t, these one-period agents solve a maximization problem where they choose the quantity of stocks purchased y_{1t} (a vector with one element for each asset) to maximize utility

$$\begin{aligned} & \max_{y_{1t}} (E_t(W_{t+1}) - \frac{1}{2} A_1 V_t(W_{t+1})) \\ W_{t+1} &= (P_{t+1} + D_{t+1} - C_{t+1})' y_{1t} + r_f(e_1 - P_t' y_{1t}), \end{aligned} \tag{1}$$

where W_{t+1} is wealth at time $t+1$, P_{t+1} is the vector of prices, and e_1 is the endowment.

The two-period investors are also mean-variance optimizers, but care about their wealth two periods ahead. For simplicity, we do not allow these two-period agents to rebalance after one period. In essence, we assume that rebalancing trades of long-term investors are relatively small and can be ignored. The utility maximization is then given by

$$\begin{aligned} & \max_{y_{2t}} (E_t(W_{t+2}) - \frac{1}{2} A_2 V_t(W_{t+2})) \\ W_{t+2} &= (P_{t+2} + D_{t+1} + D_{t+2} - C_{t+2})' y_{2t} + r_f(D_{t+1} + r_f(e_2 - P_t' y_{2t})). \end{aligned} \tag{2}$$

We assume that two one-period investors and one two-period investor enter the market in each period. Also, for simplicity, we assume that both dividends and costs are i.i.d. Then, given that demand is independent of wealth, given a fixed asset supply, and with the same type of investors entering the market each period, we obtain a stationary equilibrium where the price of each asset P_i will be constant over time. At any point in time, the market clears with new investors buying the supply of stocks minus the amount held by the two-period investor that entered the market one period ago,

$$2y_{1t} + y_{2t} = S - y_{2,t-1}, \tag{3}$$

where S is vector with supply of assets (in amount of each of the assets). Given the i.i.d. setting, we have constant demand over time, $y_{1t} = y_{1,t-1}$ and $y_{2t} = y_{2,t-1}$.

Below, we work out the equilibrium expected returns for various cases. To set the stage, we start studying the case where all investors have the same horizon. Then we allow for horizon heterogeneity, and consider two potential equilibria. In the first case (*integration*), both investors have strictly positive holdings in both assets. In the second case (*partial segmentation*), the short-term investor only invests in the asset with low transaction costs (i.e., his optimal position in the high-cost asset is equal to zero, since the transaction costs prevent this investor from buying or selling the asset).

2.1 Case 0, *homogeneity*: both investors have the same horizon

If all investors have the same one-period horizon, we obtain Acharya and Pedersen's liquidity CAPM. This can be seen as follows. The optimal demand of the investor is

$$y_1 = \frac{1}{A_1} diag(P_t)^{-1} Var(r_{t+1} - c_{t+1})^{-1} (E(1 + r_{t+1} - c_{t+1}) - r_f) \tag{4}$$

where r denotes the asset return and c the percentage costs, $c_t(i) = C_t(i)/P_t(i)$. Solving the equilibrium condition $2y_{1t} = S$, with two investors entering the market each period, gives

$$E(1 + r_{t+1}) - r_f = E(c_{t+1}) + \frac{A_1}{2}\widetilde{S}'\iota Cov(r_{t+1} - c_{t+1}, r_{m,t+1} - c_{m,t+1}) \tag{5}$$

where $\widetilde{S} = diag(P_t)S$ is dollar supply (which is constant over time given that prices are constant over time), and where $r_m = \widetilde{S}'r/\widetilde{S}'\iota$.

Alternatively, if all investors are two-period investors (with a new two-period investor entering the market each period), the Appendix shows that the optimal demand is

$$y_2 = \frac{1}{A_2}diag(P_t)^{-1}Var(r_{t+1} + r_{t+2} - c_{t+2})^{-1}(E(1 + r_{t+1} + r_{t+2} - c_{t+2}) - r_f^2) \tag{6}$$

Using the equality $Var(r_{t+1}+r_{t+2}-c_{t+2}) = Var(r_{t+1})+Var(r_{t+2}-c_{t+2})$, valid in our i.i.d. setting, and the approximation $r_f^2 \approx 2r_f - 1$, the market clearing condition $y_{2t} = S - y_{2,t-1}$ leads to the equilibrium expected returns,

$$E(1+r_{t+1}) - r_f = \frac{1}{2}E(c_{t+1}) + \frac{A_2}{4}\widetilde{S}'\iota(Cov(r_{t+1}, r_{m,t+1}) + Cov(r_{t+1} - c_{t+1}, r_{m,t+1} - c_{m,t+1})). \tag{7}$$

Comparing equilibrium expected returns in equation (7) to the one-period case in equation (5), we observe that the coefficient on expected liquidity decreases from 1 to 1/2, due to the longer horizon. In addition, the role of liquidity risk is smaller, given that the first-period return is not affected by liquidity costs.

2.2 Case 1, *integration*: both investors invest in both assets

We now turn to the case with heterogeneous horizons. We first consider the case where the optimal demands y_1 and y_2 are strictly positive, so that both investor types have positive holdings of both assets. This corresponds to a situation where the liquidity costs are sufficiently small. Using the market clearing condition (3) and optimal demands in (4) and (6), the Appendix derives the equilibrium expected returns

$$E(1 + r) - r_f = \Phi E(c) + (\lambda_1 + \lambda_2)Cov(r - c, r_m - c_m) + \lambda_2 Cov(r, r_m) \tag{8}$$

$$\Phi = \gamma_1 I - \gamma_2(V(r - c)^{-1}V(r) + I)^{-1} \tag{9}$$

where we suppress all time subscripts given the i.i.d. nature of the equilibrium and where λ_1, λ_2, γ_1, and γ_2 are scalars that are functions of the risk aversion levels and covariance matrices of returns and costs (see the Appendix). The Appendix shows that $\lambda_1 > 0$, $\lambda_2 > 0$, $\gamma_1 > 0$ and $\gamma_2 > 0$.

Equation (8) shows that for the risk premium term one gets a mixture of the net-of-cost covariance and the regular CAPM covariance. This makes intuitive sense: the presence of long-term investors implies that investors care more about regular market risk and relatively less about liquidity risk. The weights on these two covariances depend, amongst others, on the risk aversion of the one-period and two-period investors. For example, in the Appendix we show that as the long-term investors become relatively more risk averse (or short-term agents become less risk averse), the liquidity risk covariance becomes more important relative to the market covariance (formally $\frac{\lambda_1+\lambda_2}{(\lambda_1+\lambda_2)+\lambda_2} \uparrow$ as $A_2/A_1 \uparrow$). This makes intuitive sense. When long-term investors are more risk averse (or short-term investors less risk averse), the long-term investors hold a relatively smaller fraction of the supply in equilibrium, and hence the demand of the short-term investors is the predominant factor determining expected returns. Since short-term investors care more about liquidity risk, the liquidity risk premium becomes relatively more important in equilibrium.

Next, we turn to loading on expected liquidity, as defined by the matrix Φ in equation (9). This term contains two important insights. First, if there is no liquidity risk ($V(c) = 0$) we obtain $V(r-c)^{-1}V(r) = I$, and the effect of expected liquidity is the same for both assets and equal to $\gamma_1 - \frac{1}{2}\gamma_2$. The Appendix shows that $\gamma_1 - \frac{1}{2}\gamma_2 < 1$, so that the coefficient on expected liquidity is smaller than 1 (which is the coefficient in the baseline one-period model in Section 2.1), due to the presence of two-period investors that care less about expected liquidity.

If $V(c) > 0$, the coefficients on expected liquidity may vary across the two assets due to covariance between costs and returns. This reflects the fact that short-term investors care more about liquidity risk covariances than long-term investors.

For example, suppose the second asset has no liquidity risk ($V(c_2) > 0$), while the first asset has liquidity risk ($V(c_1) > 0$). In addition, suppose that for asset 1, $Cov(r_1, c_1) < 0$, so that $V(r_1 - c_1) > V(r_1)$. The first asset is then less attractive for short-term investors since high costs coincide with low returns, while this liquidity risk is less important for long-term investors. It then follows directly that the expected liquidity matrix Φ is diagonal, with the coefficient on expected liquidity for asset 2 is equal to $\Phi_{2,2} = \gamma_1 - \frac{1}{2}\gamma_2$, while for asset 1 we obtain

$$\Phi_{1,1} = \gamma_1 - \frac{1}{1 + V(r_1)/V(r_1 - c_1)}\gamma_2 < \Phi_{2,2} \tag{10}$$

It thus follows that for asset 1 the coefficient on expected liquidity is smaller than for asset 2: since the first asset is relatively less attractive for short-term investors, it will be held in equilibrium mostly by long-term investors that care less about liquidity, leading to a smaller coefficient for the expected liquidity effect. Hence, we see that higher liquidity risk may actually lead to a smaller expected liquidity premium.

2.3 Case 2, *partial segmentation*: only the long-term investor invests in both assets

We now turn to the case where the costs on one asset are so high that, in equilibrium, the one-period investors optimally invest only in the low-cost asset and have a zero position in the high-cost asset. Suppose asset 1 has higher costs than asset 2. In fact, costs are so high that, in equilibrium, $y_1(1) < 0$ (and $y_1(2) > 0$). This means that short-term investors do not want to buy asset 1. Of course, it is still possible that the investor wants to short asset 1, but this is unlikely given the high transaction costs. To see this formally, if the optimal position in asset 1 were negative (and positive for asset 2), the optimal portfolio would be

$$z_1 = \frac{1}{A_1} diag(P_t)^{-1} Var(r_{t+1} - \delta_1 c_{t+1})^{-1}(E(1 + r_{t+1} - \delta_1 c_{t+1}) - r_f) \tag{11}$$

where $\delta_1 = diag(-1, 1)$, hence δ_1 is a diagonal matrix with elements equal to 1 if the investor is long in the respective asset, and -1 if he is short (see Bongaerts, De Jong, and Driessen, 2011). If $z_1(1) < 0$, this is indeed the solution to the optimal portfolio rule, but this is unlikely if costs are high for this asset. In turn, if $z_1(1) > 0$ and $y_1(1) < 0$, it is optimal for the investor to have zero position in asset 1. We thus focus here on the case in which costs are high enough so that the investor has a zero position in asset 1.

This simplifies the optimal allocation of agent 1,

$$\widetilde{y}_1 = \begin{pmatrix} 0 \\ \frac{1}{A_1} Var(r_2 - c_2)^{-1}(E(1 + r_2 - c_2) - r_f) \end{pmatrix} \tag{12}$$

The demand of agent 2 is unchanged from above. Appendix XX derives the equilibrium expected returns,

$$\begin{aligned} E(1 + r) - r_f &= \Lambda_1^{-1}\Lambda_2 E(c) + \phi_1(Cov(r, r_m) + Cov(r - c, r_m - c_m)) + \begin{pmatrix} \phi_2 \widetilde{S}_1 \\ 0 \end{pmatrix} \qquad (13) \\ \Lambda_1 &= A_1^{-1} \begin{pmatrix} 0 & 0 \\ 0 & V(r_2 - c_2)^{-1} \end{pmatrix} + 2A_2^{-1}(V(r) + V(r - c))^{-1}) \\ \Lambda_2 &= A_1^{-1} \begin{pmatrix} 0 & 0 \\ 0 & V(r_2 - c_2)^{-1} \end{pmatrix} + A_2^{-1}(V(r) + V(r - c))^{-1}) \end{aligned}$$

where the parameters ϕ_i are scalars, and the Appendix shows that $\phi_1 > 0$ and $\phi_2 > 0$.

This shows that we get two deviations from the case of homogeneity of investors. First, the effect of the liquidity risk covariances is smaller (relative to the market covariance). Second, we get a segmentation result. The expected return on the first asset is higher by an extra term that reflects the fact that only a subset of the investors holds this asset. This is in the spirit of the international asset pricing literature (e.g. de Jong and de Roon, 2005), where segmentation also leads to additional effects on expected returns, that depend on the size of the supply of the segmented asset ($\widetilde{S}_1$). The Appendix shows that the coefficient on the segmented supply, ϕ_2, increases with the risk-aversion of long-term investors A_2, since these are the investors that have to hold this asset in equilibrium.

Then we turn to the expected liquidity coefficients, $\Lambda_1^{-1}\Lambda_2$. To obtain some intuition, consider the case where there is zero covariance between returns on the two assets (both before and after costs). In this case, in the Appendix we show that

$$\Lambda_1^{-1}\Lambda_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1+\eta}{2+\eta} \end{pmatrix} \tag{14}$$

with $\eta = \frac{A_2}{A_1} \frac{V(r_2 - c_2) + V(r_2)}{V(r_2 - c_2)} > 0$. This reveals several interesting effects. First, if we neglect the covariance terms, we see that the coefficient on expected liquidity is larger for the low-cost asset 2. Thus if we graph the relation between expected returns and expected costs, we get a piecewise linear and concave relation, like Amihud and Mendelson (1986).

Intuitively, when costs on an asset are too high, short-term investors drop out and only long-term investors invest in the asset. Given that long-term investors care less about liquidity, the effect of liquidity on expected returns is smaller. In this two-period, two-asset example, the coefficient is equal to 1/2 for asset 1, because the holding period is two periods.

Finally, note that Amihud and Mendelson (1986) find that long-term investors only invest in high-cost assets, and not in the low-cost assets. This is because they assume risk-neutrality. In our model with risk averse agents long-term investors will diversify and invest in low-cost assets as well.

3 A Multi-Period Liquidity-CAPM

The baseline model with two assets and two investors can be generalized to a setting with many assets and many investors with heterogenous investment horizons. This general framework is more realistic and, most importantly, is suitable for empirical estimation. Specifically, we model $j = 0, 1, \ldots, N$ classes of investors with distinct investment horizons $h_0, h_1, \ldots, h_N$. Each period, a fixed quantity $Q_j > 0$ of type j investors enters the market. We set $h_0 = 1$ and $Q_0 > 0$, thus assuming that there are at least some 1-period investors. For ease of notation we will write $R_t = 1 + r_t$.

Appendix YY shows that this setup generates the following equilibrium expected returns:

$$\begin{aligned}
\mathbb{E}\left[R_{t+1}\right] - R_f = &\left(I + \sum_{j=1}^{N} h_j^2 \frac{A_0/Q_0}{A_j/Q_j} \mathrm{Var}\left(R_{t+1} - c_{t+1}\right) \mathrm{Var}\left(\sum_{k=1}^{h_j} R_{t+k} - c_{t+h_j} \right)^{-1} \right)^{-1} \mathbb{E}\left[c_{t+1}\right] \\
&- \frac{A_0}{Q_0} \left(I + \sum_{j=1}^{N} h_j^2 \frac{A_0/Q_0}{A_j/Q_j} \mathrm{Var}\left(R_{t+1} - c_{t+1}\right) \mathrm{Var}\left(\sum_{k=1}^{h_j} R_{t+k} - c_{t+h_j} \right)^{-1} \right)^{-1} \\
&\times \widetilde{S}_t' \iota \mathrm{Cov}\left(R_{t+1} - c_{t+1}, R_{t+1}^m - c_{t+1}^m\right) \\
&- \left(I + \sum_{j=1}^{N} h_j^2 \frac{A_0/Q_0}{A_j/Q_j} \mathrm{Var}\left(R_{t+1} - c_{t+1}\right) \mathrm{Var}\left(\sum_{k=1}^{h_j} R_{t+k} - c_{t+h_j} \right)^{-1} \right)^{-1} \\
&\times \sum_{j=1}^{N} h_j \frac{A_0/Q_0}{A_j/Q_j} \mathrm{Var}\left(R_{t+1} - c_{t+1}\right) \mathrm{Var}\left(\sum_{k=1}^{h_j} R_{t+k} - c_{t+h_j} \right)^{-1} \\
&\times \left(\mathbb{E}\left[c_{t+h_j}\right] + (h_j - 1) + R_f \left(R_f^{h_j - 1} - h_j \right) \right). \qquad (15)
\end{aligned}$$

Equation (15) corresponds to the case of integration described in the basic version of the model in Section 2.2. We now introduce in this general framework the feature of segmentation, i.e. the possibility that some classes of investors do not hold some assets because the associated trading costs are too high relative to the expected return over the investment horizon. To this end, we introduce sets D_j ($j = 0, \ldots, N$) that are subsets of $1, \ldots, K$, where K is the number of tradable assets. The set D_j represents the set of investable assets for the class of investors j. For example, a short horizon investor rules out certain assets *a-priori* because the associated transaction costs are too large.

Without loss of generality we assume that for some j it holds that $D_j = \{1, \ldots, K\}$. For ease of exposition, we assume that $D_j \subset D_{j+1}$ for each $j < N$. This then implies that $D_N = \{1, \ldots, K\}$.

We denote with A_{D_j} any $K \times K$ matrix A that is equal to the matrix A without the rows and columns whose indices are not included in D_j. Appendix YY shows that, in this setting, equilibrium excess returns are defined by the following equation:

$$
\begin{aligned}
\mathbb{E}\left[R_{t+1}\right] - R_f = & \left(V_{0,t}^{D_0} + \sum_{j=1}^{N} h_j V_{j,t}^{D_j} \right)^{-1} \mathbb{E}\left[c_{t+1}\right] \\
& + \frac{A_0 \widetilde{S}_t' \iota}{Q_0} \left(V_{0,t}^{D_0} + \sum_{j=1}^{N} h_j V_{j,t}^{D_j} \right)^{-1} \operatorname{Cov}\left(R_{t+1} - c_{t+1}, R_{t+1}^m - c_{t+1}^m\right) \\
& + \left(V_{0,t}^{D_0} + \sum_{j=1}^{N} h_j V_{j,t}^{D_j} \right)^{-1} \sum_{j=1}^{N} V_{j,t}^{D_j} \left(\mathbb{E}\left[c_{t+h_j}\right] + (h_j - 1) + R_f \left(R_f^{h_j - 1} - h_j \right) \right),
\end{aligned}
\tag{16}
$$

where

$$
V_{j,t}^{D_j} = \frac{h_j A_0 Q_j}{A_j Q_0} \operatorname{Var}\left(R_{t+1} - c_{t+1}\right) \operatorname{Var}\left(\sum_{k=1}^{h_j} R_{t+k} - c_{t+h_j} \right)_{D_j,p}^{-1}.
$$

4 Empirical Methodology

The theoretical setup developed in the previous Sections generates a setting with segments of assets held by different investors. Specifically, low-cost assets held by all investors, medium-cost assets held by investors with medium-term or long-term horizons, and high-cost assets only held by long-term investors. Stocks in the different segments will have expected equilibrium returns with different compositions of liquidity premiums, standard CAPM risk premiums, and segmentation effects. In this Section, we start explaining the details of our estimation methodology. We then discuss alternative approaches for a robust computation of standard errors.

4.1 General Setting

We use a Generalized Method of Moments (GMM) methodology to estimate the equilibrium condition in the general case, as defined by equation (15). The key estimated parameters are the risk aversion coefficients of the different classes of investors or, more generally, risk aversion divided by the number of agents per holding period. More specifically, we estimate $\gamma_0 = A_0/Q_0$ and $\gamma_j = (A_0/Q_0)/(A_j/Q_j)$. We use the moment conditions $\mathbb{E}\left[g(\gamma)\right]=0$, where the elementary zero functions $g(\gamma)$ are given by

$$\begin{aligned}
g(\gamma) = {} & \mathbb{E}\left[R_{t+1}\right] - R_f - \left(I + \sum_{j=1}^{N} h_j^2 \gamma_j \mathrm{Var}\left(R_{t+1}-c_{t+1}\right) \mathrm{Var}\left(\sum_{k=1}^{h_j} R_{t+k} - c_{t+h_j} \right)^{-1} \right)^{-1} \mathbb{E}\left[c_{t+1}\right] \\
& - \gamma_0 \left(I + \sum_{j=1}^{N} h_j^2 \gamma_j \mathrm{Var}\left(R_{t+1}-c_{t+1}\right) \mathrm{Var}\left(\sum_{k=1}^{h_j} R_{t+k} - c_{t+h_j} \right)^{-1} \right)^{-1} \widetilde{S}_t' \iota \mathrm{Cov}\left(R_{t+1}-c_{t+1}, R_{t+1}^m - \right. \\
& - \left(I + \sum_{j=1}^{N} h_j^2 \gamma_j \mathrm{Var}\left(R_{t+1}-c_{t+1}\right) \mathrm{Var}\left(\sum_{k=1}^{h_j} R_{t+k} - c_{t+h_j} \right)^{-1} \right)^{-1} \\
& \times \sum_{j=1}^{N} h_j \gamma_j \mathrm{Var}\left(R_{t+1}-c_{t+1}\right) \mathrm{Var}\left(\sum_{k=1}^{h_j} R_{t+k} - c_{t+h_j} \right)^{-1} \left(\mathbb{E}\left[c_{t+h_j}\right] + (h_j - 1) + R_f \left(R_f^{h_j-1} - h_j \right) \right)
\end{aligned} \tag{17}$$

A complication in the estimation is that the expected returns and covariances included in these equations are themselves unknown and need to be estimated. We thus use the delta method and estimate the model in two stages: first, expected returns and covariance, then the model parameters.

We denote the required moments by ψ, the corresponding elementary zero functions in the first stage by $g_1(\psi)$, and the second stage by $g_2(\psi,\gamma)$, where we make explicit the dependence on ψ. We derive the standard errors as follows. Since there is sampling uncertainty in the second stage as the estimator of ψ is a consistent estimator for the population moments, we write the second-stage moment conditions as

$$g_2(\psi,\gamma)=0. \tag{18}$$

Let

$$g_{1T}(x,\psi)=\frac{1}{T}\sum_{t=1}^{T} g_1(x_t,\psi), \tag{19}$$

where the dependence on the observations x_t will be suppressed in what follows. Looking at the first stage, we have

$$\sqrt{T}\left(\widehat{\psi}-\psi\right) \xrightarrow{d} \mathcal{N}\left(0, G_{1\psi}^{-1} S_\psi \left(G_{1\psi}'\right)^{-1}\right), \tag{20}$$

where

$$G_{1\psi}(\psi)=\frac{\partial g_{1T}(\psi)}{\partial \psi} \qquad \text{and} \qquad S_\psi = \lim_{T\to\infty} \mathrm{Var}\left(\sqrt{T} g_{1T}(\psi)\right).$$

We can now use the delta method to find the standard errors for $\widehat{\gamma}$, as $\widehat{\gamma}$ is implicitly given as the solution of the elementary zero function in the second stage or, equivalently, as the solution of the GMM minimization problem.

Appendix ZZ derives the following result

$$\sqrt{T}\left(\widehat{\gamma}-\gamma_0\right) \xrightarrow{d} \mathcal{N}\left(0, \left(G_{2\gamma}' G_{2\gamma}\right)^{-1} G_{2\gamma}' G_{2\psi} G_{1\psi}^{-1} S_\psi \left(G_{1\psi}'\right)^{-1} G_{2\psi}' G_{2\gamma} \left(G_{2\gamma}' G_{2\gamma}\right)^{-1}\right), \tag{21}$$

where G_{ix} is the gradient of the elementary zero function of stage i with respect to the estimated parameter x. This result allows us to compute standard errors for the γ estimates taking into account the pre-estimation of the various moments.

4.2 Shrinkage

A potential issue could arise given that the estimation requires inversion of the estimated covariance matrix of returns. In finite samples, this inversion may lead to extreme long-short position. Michaud (1989) points out that matrix inversion maximizes the effects of errors in the input assumptions. Consistent with this, Britten-Jones (1999) finds the sampling error of the weights of mean-variance efficient portfolios to be very large.

As a first step towards addressing this issue, we use the i.i.d. assumption to rewrite part of the moment conditions as follows

$$\begin{aligned}
&\mathrm{Var}\,(R_{t+1} - c_{t+1})\mathrm{Var}\left(\sum_{k=1}^{h_j} R_{t+k} - c_{t+h_j}\right)^{-1} \\
&\qquad = \mathrm{Var}\,(R_{t+1} - c_{t+1})\,\mathrm{Var}\left(\sum_{k=1}^{h_j-1} R_{t+k} + R_{t+h_j} - c_{t+h_j}\right)^{-1} \\
&\qquad = \mathrm{Var}\,(R_{t+1} - c_{t+1})\left(\mathrm{Var}\left(\sum_{k=1}^{h_j-1} R_{t+k}\right) + \mathrm{Var}\left(R_{t+h_j} - c_{t+h_j}\right)\right)^{-1} \\
&\qquad = \mathrm{Var}\,(R_{t+1} - c_{t+1})\left((h_j - 1)\mathrm{Var}\,(R_{t+1}) + \mathrm{Var}\,(R_{t+1} - c_{t+1})\right)^{-1}. \qquad (22)
\end{aligned}$$

For the short horizon covariance matrix $\mathrm{Var}\,(R_{t+1})$ (and possibly also for $\mathrm{Var}\,(R_{t+1} - c_{t+1})$) we use a shrinkage estimator for greater stability (see Ledoit and Wolf, 2003).

Fan, Fan, and Lv (2008) consider a factor model as a conditioning approach and demonstrate that this model provides a better conditioned alternative to the fully-estimated covariance matrix of stock returns.

Our shrinkage estimator is based on the Sharpe (1963) single index model (a factor implementation of the standard CAPM). Let β denote the vector of CAPM betas corresponding to each portfolio and let Δ denote the diagonal matrix of residual variances corresponding to each factor regression. The single index model then can be used to estimate $\mathrm{Var}\,(R_{t+1})$ by

$$F = \mathrm{Var}\left(R^m_{t+1}\right)\beta\beta' + \Delta. \qquad (23)$$

The shrinkage estimator is given by

$$\widehat{\mathrm{Var}}\,(R_{t+1}) = \frac{k}{T}F + \left(1 - \frac{k}{T}\right)\mathrm{Var}\,(R_{t+1}), \qquad (24)$$

where k is (a consistent estimator for) the optimal shrinkage constant as derived by Ledoit and Wolf (2003). To obtain a shrinkage estimator for $\mathrm{Var}\,(R_{t+1} - c_{t+1})$ we can follow a similar procedure, using the Acharya and Pedersen (2005) liquidity CAPM instead of the standard CAPM.

4.3 Bootstrapped Standard Errors

5 Data

We use daily stock return and volume data from CRSP from 1962 until 2004 for all common shares listed on NYSE and AMEX. As our empirical measures of liquidity rely on volume, we do not include Nasdaq since the volume data includes interdealer trades (and only starts in 1982). Overall, we consider a number of stocks ranging from 1056 to 3358, depending on the month. To correct for survivorship bias, we adjust the returns for stock delisting (see Shumway, 1997; Acharya and Pedersen, 2005). Some descriptive statistics are given in Table 1.

The relative illiquidity cost is computed as in Acharya and Pedersen (2005). The starting point is the Amihud (2002) illiquidity measure, which is defined as

$$ILLIQ_t^j = \frac{1}{D_t^j} \sum_{d=1}^{D_t^j} \frac{|R_{td}^j|}{V_{td}^j} \tag{25}$$

for stock j in month t, where D_t^j denotes the number of observations available in month t, R_{td}^j and V_{td}^j denote the volume in millions of dollars on day d in month t, respectively.

We follow Acharya and Pedersen (2005) and define a normalized measure of illiquidity that deals with non-stationarity and is a direct measure of trading costs, consistent with the model specification. The normalized illiquidity measure can be interpreted as the dollar cost per dollar invested and is defined by

$$c_t^j = \min \left\{ 0.25 + 0.30 ILLIQ_t^j P_{t-1}^m, 30.00 \right\}, \tag{26}$$

where P_{t-1}^m is equal to the market capitalization of the market portfolio at the end of month $t-1$ divided by the value at the end of July 1962. The product with P_{t-1}^m makes the cost series c_t^j relatively stationary and the coefficients 0.30 and 0.25 are chosen as in Acharya and Pedersen (2005) to match approximately the level and variance of c_t^j for the size portfolios to those of the effective half spread reported by Chalmers and Kadlec (1998). The value of normalized liquidity c_t^j is capped at 30% to make sure the empirical results are not driven by outliers.

Turnover is computed as dollar volume divided by market capitalization. As the monthly turnover series contains some outliers (e.g. exchange traded funds with relatively low market capitalization), we censor the turnover series at 500%. This affects 1023 data points.

We obtain the book-to-market ratio using fiscal year-end balance sheet data from COMPUSTAT in the same manner as Ang and Chen (2002). They follow Fama and French (1993) in defining the book value of a firm as the sum of common stockholders' equity, deferred taxes, and investment credit minus the book value of preferred stocks. The ratio is obtained by dividing the book value by the fiscal year-end market value.

We construct the market portfolio on a monthly basis and only use stocks that have a price on the first trading day of the corresponding month between \$5 and \$1000. We include only stocks that have at least 15 observations of return and volume during the month.

We construct 25 illiquidity portfolios, 25 illiquidity variation portfolios, and 25 book-to-market and size portfolios, similarly to Acharya and Pedersen (2005). The portfolios are formed on an annual basis. For these portfolios, we require again for the stock price on the first trading day of the corresponding month to be between \$5 and \$1000. For the illiquidity and illiquidity variation portfolios, we require to have at least 100 observations of the illiquidity measure in the previous year.

6 Empirical Results

The estimation was performed for the period 1964–2004. Two classes of investors were used. The first class has an investment horizon of one month, the second class an investment horizon of 60 months (5 years; twice the average holding period implied by the turnover of the stocks in our sample). The estimated parameters are $\gamma_0 = A_0/Q_0$ and $\gamma_j = (A_0/Q_0)/(A_j/Q_j)$, as well as a constant term α and a coefficient κ for the cost term. $\widetilde{S}_t'\iota$ was assumed to be constant and absorbed into the γ_0. We considered both the sample moment estimator as well as the shrinkage estimator for the covariance matrices. It turned out that the sample moment yields more precise estimates, which is most likely due to our relatively large time dimension. In fact, we have almost 500 time series observations corresponding to each of the 25 cross sectional observations. Therefore, the reported results do not use the shrinkage procedure.

The results in Table 2 show that the first specification of the model improves the R^2 obtained by Acharya and Pedersen (2005) by about 5%. If we assume that risk aversion is constant across investor classes (i.e. that $A_0 = A_1$), we can make inferences about the number of investors in each class. We can do this through examining $\gamma_0/\gamma_1 = A_1/Q_1$ and comparing it to A_0/Q_0. This amounts to computing Q_1/Q_0, which is equal to $1/\gamma_1$. Note that including $\widetilde{S}_t'\iota$ in γ_0 does not influence our comparison. The results for the first and fourth specification respectively indicate that there are about 48 and 6 times as many long horizon investors as there are short horizon investors. This is consistent across all specifications: the number of long term investors is much larger than the number of short term investors.

From Figure 2 and Figure 3 we see that the covariance term provides by far the largest contribution to the expected excess returns. The holding period variance term also provides a large contribution across all portfolios. The absolute contribution grows with the level of illiquidity and the relative contribution is larger for the most extreme portfolios. The impact of the cost term increases with the level of illiquidity of the portfolio. Note that this does not follow by definition, as the expected cost is premultiplied by a term depending among other things on γ_1.

7 Conclusions

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8 Appendix

8.1 Two-period two-asset model

In this Appendix we provide derivations for several equations in Section 2.

Optimal demand two-period investors

Two-period agents solve

$$\begin{aligned} & \max_{y_2}(E_t(W_{t+2}) - \frac{1}{2}A_2 V_t(W_{t+2})) \qquad (27) \\ W_{t+2} &= (P_{t+2} + D_{t+1} + D_{t+2} - C_{t+2})' y_2 + r_f(D_{t+1} + r_f(e_2 - P_t' y_2)) \end{aligned}$$

The solution to this problem is

$$y_2 = \frac{1}{A_2} Var_t(P_{t+2} + D_{t+1} + D_{t+2} - C_{t+2})^{-1}(E_t(P_{t+2} + D_{t+1} + D_{t+2} - C_{t+2}) - r_f^2 P_t) \qquad (28)$$

or

$$y_2 = \frac{1}{A_2} diag(P_t)^{-1} Var_t(1 + r_{t+1} + \frac{P_{t+1}}{P_t} r_{t+2} - \frac{P_{t+2}}{P_t} c_{t+2})^{-1}(E_t(1 + r_{t+1} + \frac{P_{t+1}}{P_t} r_{t+2} - \frac{P_{t+2}}{P_t} c_{t+2}) - r_f^2) \qquad (29)$$

In equilibrium, prices P_t are constant over time, and we obtain equation (6).

Equilibrium in case of integration

Filling in the optimal demands into the equilibrium condition $2y_1 + y_2 = S - y_2$ and multiplying both sides by $Var(r_{t+1} - c_{t+1})$ gives

$$\begin{aligned} & \frac{2}{A_1}(E(1 + r_{t+1} - c_{t+1}) - r_f) + \qquad (30) \\ & \frac{2}{A_2} Var(r_{t+1} - c_{t+1}) Var(r_{t+1} + r_{t+2} - c_{t+2})^{-1}(E(1 + r_{t+1} + r_{t+2} - c_{t+2}) - r_f^2) \\ = \; & Var(r_{t+1} - c_{t+1})\widetilde{S} = \widetilde{S}'\iota Cov(r_{t+1} - c_{t+1}, r_{m,t+1} - c_{m,t+1}) \end{aligned}$$

Dropping time subscripts and approximating r_f^2 by $2r_f - 1$, this equilibrium condition can be rewritten as

$$\begin{aligned} E(1 + r) - r_f &= \left(A_1^{-1} V(r - c)^{-1} + 2A_2^{-1}(V(r) + V(r - c))^{-1})\right)^{-1} \widetilde{S}/2 + \qquad (31) \\ & \left(A_1^{-1} V(r - c)^{-1} + 2A_2^{-1}(V(r) + V(r - c))^{-1})\right)^{-1} \cdot \\ & \left(A_1^{-1} V(r - c)^{-1} + A_2^{-1}(V(r) + V(r - c))^{-1})\right) E(c) \end{aligned}$$

In the two-asset case, the term $\left(A_1^{-1} V(r - c)^{-1} + 2A_2^{-1}(V(r) + V(r - c))^{-1})\right)^{-1} \widetilde{S}/2$ can be written

$$\frac{1}{d_0}(\frac{1}{d_1} + \frac{1}{d_2})\frac{\widetilde{S}'\iota}{2} Cov(r - c, r_m - c_m) + \frac{1}{d_0 d_2}\frac{\widetilde{S}'\iota}{2} Cov(r, r_m) \qquad (32)$$

with

$$\begin{aligned} d_0 &= \det(A_1^{-1} V(r - c)^{-1} + 2A_2^{-1}(V(r) + V(r - c))^{-1}) \qquad (33) \\ d_1 &= A_1 \det(V(r - c)) \\ d_2 &= \frac{1}{2} A_2 \det(V(r) + V(r - c)) \end{aligned}$$

The equilibrium terms for $Cov(r, r_m)$ and $Cov(r-c, r_m-c_m)$ in equation (8) then follow directly with $\lambda_1 = \frac{1}{d_0 d_1}\frac{\widetilde{S}'\iota}{2}$ and $\lambda_2 = \frac{1}{d_0 d_2}\frac{\widetilde{S}'\iota}{2}$. Both λ_1 and λ_2 are positive because the determinants of covariance matrices are positive. It is easy to see that the liquidity premium, relative to the total risk premium, $\frac{\lambda_1+\lambda_2}{(\lambda_1+\lambda_2)+\lambda_2}$, can be written as $1-\frac{1}{d_2/d_1+2}$ which is increasing in A_2/A_1.

Next we turn to the expected liquidity effect

$$\begin{aligned}&\left(A_1^{-1}V(r-c)^{-1}+2A_2^{-1}(V(r)+V(r-c))^{-1})\right)^{-1}\cdot\\&\left(A_1^{-1}V(r-c)^{-1}+A_2^{-1}(V(r)+V(r-c))^{-1})\right)E(c)\end{aligned}\tag{34}$$

which in the two-asset case can be rewritten as

$$\begin{aligned}&I\cdot E(c)-\left(\frac{1}{d_0}(\frac{1}{d_1}+\frac{1}{d_2})V(r-c)+\frac{1}{d_0}\frac{1}{d_2}V(r)\right)\cdot\\&\left(A_2^{-1}(V(r)+V(r-c))^{-1})\right)E(c)\end{aligned}\tag{35}$$

which can be simplified into

$$(1-\frac{1}{d_0 d_2})I\cdot E(c)-\frac{1}{d_0 d_1 A_2}(V(r-c)^{-1}V(r)+I)^{-1}E(c)\tag{36}$$

With $\gamma_1 = 1-\frac{1}{d_0 d_2}$ and $\gamma_2 = \frac{1}{d_0 d_1 A_2}$ we then obtain the expression for Φ in equation (8). Finally, we show that $\gamma_1 - \frac{1}{2}\gamma_2 < 1$. This inequality follows directly as all determinants d_i are positive.

Equilibrium in case of segmentation

In this case the equilibrium condition is

$$\begin{aligned}&\left(A_1^{-1}\begin{pmatrix}0 & 0\\ 0 & V(r_2-c_2)^{-1}\end{pmatrix}+2A_2^{-1}(V(r)+V(r-c))^{-1})\right)(E(1+r)-r_f)=\\&\widetilde{S}/2+\left(A_1^{-1}\begin{pmatrix}0 & 0\\ 0 & V(r_2-c_2)^{-1}\end{pmatrix}+A_2^{-1}(V(r)+V(r-c))^{-1})\right)E(c)\end{aligned}\tag{37}$$

or, in short, $B\left(E(1+r)-r_f\right)=\widetilde{S}/2+CE(c)$. First, the liquidity risk implications follow from working out the terms in the matrix B,

$$B^{-1}\overline{S}=\phi_1(Cov(r,r_m)+Cov(r-c,r_m-c_m))+\begin{pmatrix}\phi_2\widetilde{S}_1\\ 0\end{pmatrix}\tag{38}$$

where ϕ_1 and ϕ_2 are scalars, with

$$\phi_1 = \frac{1}{A_2 d_3 d_4}\widetilde{S}'\iota > 0\tag{39}$$

$$\phi_2 = \frac{1}{2A_1 d_4 V(r_2-c_2)} > 0\tag{40}$$

$$d_3 = \det(V(r)+V(r-c))\tag{41}$$

$$d_4 = \det(B)\tag{42}$$

From the definition of d_2 it directly follows that ϕ_2 is increasing in A_2. Then we turn to the expected liquidity coefficients, $B^{-1}C$, and we get

$$B^{-1}C = \frac{2}{A_2 d_1 d_2} \left(\begin{array}{c} \frac{1}{A_2} I + \\ \frac{1}{A_1 V(r_2 - c_2)} \left(\begin{array}{cc} \frac{1}{2}(V(r_2) + V(r_2 - c_2)) & \frac{1}{2}(Cov(r_{1,} r_2) + Cov(r_1 - c_1, r_2 - c_2)) \\ 0 & (V(r_2) + V(r_2 - c_2)) \end{array} \right) \end{array} \right) \tag{43}$$

If the covariances are zero, this simplifies to (after some algebra)

$$\left(\begin{array}{cc} \frac{1}{2} & 0 \\ 0 & \frac{1+\eta}{2+\eta} \end{array} \right) \tag{44}$$

with $\eta = \frac{A_2}{A_1} \frac{V(r_2 - c_2) + V(r_2)}{V(r_2 - c_2)} > 0$.

8.2 Multi-period multi-asset model

Equilibrium in case of integration

We derive here the equilibrium for equation (15):

Equilibrium in case of segmentation

Here we start defining some useful notation.

Therefore A_{D_j} is a $|D_j| \times |D_j|$ matrix, where $|\cdot|$ denotes the cardinality of a set. As it will be used frequently, we introduce the notation $A^{-1}_{D_j,p}$ for the inverse of A_{D_j} with zeros inserted at the locations where rows and columns of A were removed, so that $A^{-1}_{D_j,p}$ is a $K \times K$ matrix. Note that formally, for $A^{-1}_{D_j,p}$ to be well-defined, it is not necessary that A be invertible. It is only required that A_{D_j} be invertible.

For example, let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

and let $D_j = \{1, 3\}$. Then

$$A_{D_j} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$$

and $\det A_{D_j} = 1$, which implies $A^{-1}_{D_j} = \operatorname{adj} A_{D_j}$, so that

$$A^{-1}_{D_j} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}.$$

It follows that

$$A^{-1}_{D_j,p} = \begin{bmatrix} 7 & 0 & -2 \\ 0 & 0 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

8.3 Estimation Methodology: Obtaining Standard Errors

We start with the elementary zero function in the second sateg

$$g_2(\widehat{\psi}, \gamma) = 0, \tag{45}$$

or, equivalently, with the GMM minimization problem

$$\min_{\gamma} \; g_2(\widehat{\psi}, \gamma)' g_2(\widehat{\psi}, \gamma), \tag{46}$$

as the solution of

$$2G_{2\gamma}(\widehat{\psi}, \gamma)' g_2(\widehat{\psi}, \gamma) = 0, \tag{47}$$

where

$$G_{2\gamma}(\psi, \gamma) = \frac{\partial g_2(\psi, \gamma)}{\partial \gamma}. \tag{48}$$

Dividing both sides of (47) by 2 and evaluating at $\widehat{\gamma}$, we may write

$$G_{2\gamma}(\widehat{\psi}, \widehat{\gamma})' g_2(\widehat{\psi}, \gamma_0) + G_{2\gamma}(\widehat{\psi}, \widehat{\gamma})' \Big(g_2(\widehat{\psi}, \widehat{\gamma}) - g_2(\widehat{\psi}, \gamma_0) \Big) = 0. \tag{49}$$

Next, we expand $g_2(\widehat{\psi}, \widehat{\gamma})$ around γ_0:

$$g_2(\widehat{\psi}, \widehat{\gamma}) - g_2(\widehat{\psi}, \gamma_0) \approx G_{2\gamma}(\widehat{\psi}, \widehat{\gamma}) \left(\widehat{\gamma} - \gamma_0 \right). \tag{50}$$

It follows that

$$G_{2\gamma}(\widehat{\psi}, \widehat{\gamma})' g_2(\widehat{\psi}, \gamma_0) + G_{2\gamma}(\widehat{\psi}, \widehat{\gamma})' G_{2\gamma}(\widehat{\psi}, \widehat{\gamma}) \left(\widehat{\gamma} - \gamma_0 \right) = 0. \tag{51}$$

We now expand $g_2(\widehat{\psi}, \gamma_0)$ around ψ_0 and use the fact that $g_2(\psi_0, \gamma_0) = 0$:

$$g_2(\widehat{\psi}, \gamma_0) \approx G_{2\psi}(\widehat{\psi}, \widehat{\gamma}) \Big(\widehat{\psi} - \psi_0 \Big), \tag{52}$$

where

$$G_{2\psi}(\psi, \gamma) = \frac{\partial g_2(\psi, \gamma)}{\partial \psi}. \tag{53}$$

Hence

$$G_{2\gamma}(\widehat{\psi}, \widehat{\gamma})' G_{2\gamma}(\widehat{\psi}, \widehat{\gamma}) \left(\widehat{\gamma} - \gamma_0 \right) = -G_{2\gamma}(\widehat{\psi}, \widehat{\gamma})' G_{2\psi}(\widehat{\psi}, \widehat{\gamma}) \Big(\widehat{\psi} - \psi_0 \Big). \tag{54}$$

Using this result we obtain

$$\sqrt{T} \left(\widehat{\gamma} - \gamma_0 \right) \approx - \Big(G_{2\gamma}(\widehat{\psi}, \widehat{\gamma})' G_{2\gamma}(\widehat{\psi}, \widehat{\gamma}) \Big)^{-1} G_{2\gamma}(\widehat{\psi}, \widehat{\gamma})' G_{2\psi}(\widehat{\psi}, \widehat{\gamma}) \sqrt{T} \Big(\widehat{\psi} - \psi_0 \Big). \tag{55}$$

It follows that

$$\sqrt{T} \left(\widehat{\gamma} - \gamma_0 \right) \xrightarrow{d} \mathcal{N} \Big(0, \left(G'_{2\gamma} G_{2\gamma} \right)^{-1} G'_{2\gamma} G_{2\psi} G_{1\psi}^{-1} S_{\psi} \left(G'_{1\psi} \right)^{-1} G'_{2\psi} G_{2\gamma} \left(G'_{2\gamma} G_{2\gamma} \right)^{-1} \Big). \tag{56}$$

This result allows us to compute standard errors for the γ estimates taking into account the pre-estimation of the various moments.

Table 1: Descriptive statistics.

This table shows some descriptive statistics pertaining to the data that were used to estimate the liquidity-holding period model. The data used are monthly data corresponding to 25 value-weighted portfolios during the period 1964–2004.

Portfolio	$\mathbb{E}[c_{t+1}]$	$\mathbb{E}[R_{t+1}] - R_f$	$\mathrm{Var}(R_{t+1})$
1	0.0025	0.0038	0.0419
2	0.0026	0.0041	0.0456
3	0.0026	0.0046	0.0450
4	0.0027	0.0058	0.0457
5	0.0028	0.0061	0.0472
6	0.0029	0.0056	0.0465
7	0.0030	0.0060	0.0471
8	0.0031	0.0057	0.0481
9	0.0033	0.0059	0.0481
10	0.0035	0.0064	0.0472
11	0.0038	0.0071	0.0505
12	0.0042	0.0058	0.0473
13	0.0046	0.0066	0.0477
14	0.0050	0.0076	0.0495
15	0.0057	0.0071	0.0500
16	0.0065	0.0070	0.0494
17	0.0076	0.0086	0.0502
18	0.0087	0.0073	0.0500
19	0.0103	0.0090	0.0519
20	0.0136	0.0067	0.0526
21	0.0162	0.0082	0.0537
22	0.0208	0.0098	0.0537
23	0.0285	0.0089	0.0556
24	0.0444	0.0085	0.0556
25	0.0761	0.0099	0.0626

Table 2: Illiquidity portfolio regressions.

This table shows the results from estimation of the liquidity-holding period model. The estimates are based on monthly data corresponding to 25 value-weighted portfolios during the period 1964–2004. An equal-weighted market portfolio was used. The liquidity-holding period model specifications are special cases of the relation

$$
\begin{aligned}
\mathbb{E}\left[R_{t+1}\right]-R_f=\alpha+\kappa&\left(I+\sum_{j=1}^{N} h_j^2\gamma_j \operatorname{Var}\left(R_{t+1}-c_{t+1}\right)\operatorname{Var}\left(\sum_{k=1}^{h_j} R_{t+k}-c_{t+h_j}\right)^{-1}\right)^{-1}\mathbb{E}\left[c_{t+1}\right]\\
&-\gamma_0\left(I+\sum_{j=1}^{N} h_j^2\gamma_j \operatorname{Var}\left(R_{t+1}-c_{t+1}\right)\operatorname{Var}\left(\sum_{k=1}^{h_j} R_{t+k}-c_{t+h_j}\right)^{-1}\right)^{-1}\\
&\times\widetilde{S}_t'\iota\operatorname{Cov}\left(R_{t+1}-c_{t+1},R_{t+1}^m-c_{t+1}^m\right)\\
&-\left(I+\sum_{j=1}^{N} h_j^2\gamma_j \operatorname{Var}\left(R_{t+1}-c_{t+1}\right)\operatorname{Var}\left(\sum_{k=1}^{h_j} R_{t+k}-c_{t+h_j}\right)^{-1}\right)^{-1}\\
&\times\sum_{j=1}^{N} h_j\gamma_j \operatorname{Var}\left(R_{t+1}-c_{t+1}\right)\operatorname{Var}\left(\sum_{k=1}^{h_j} R_{t+k}-c_{t+h_j}\right)^{-1}\\
&\times\left(\mathbb{E}\left[c_{t+h_j}\right]+(h_j-1)+R_f\left(R_f^{h_j-1}-h_j\right)\right),
\end{aligned}
$$

where $N=1$, $h_0=1$, and $h_1=60$. For each coefficient the t-statistic is given in parentheses. The pseudo-R^2 is reported in the rightmost column.

	γ_0	γ_1	α	κ	R^2
1	12.89	0.0207	-0.0073	0.1386	0.7904
	(0.0737)	(0.0436)	(-0.4088)	(0.0502)	
2	91.94	0.2449	-0.0077		0.7876
	(0.1546)	(0.1599)	(-0.6423)		
3	2.694	0		0.0414	0.6954
	(0.3359)	(0)		(0.1143)	
4	29.70	0.1795			0.6750
	(0.5155)	(0.5343)			

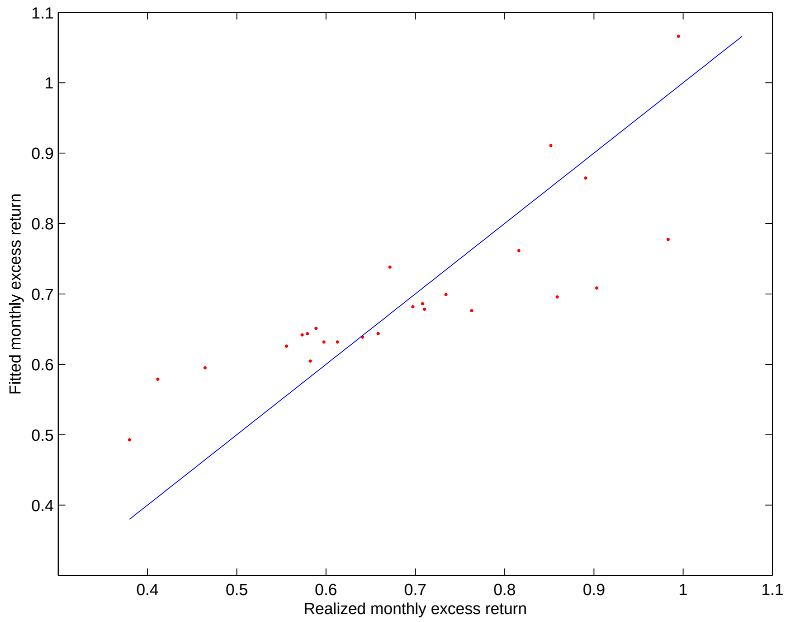

Figure 1: Fitted excess returns vs. realized excess returns.

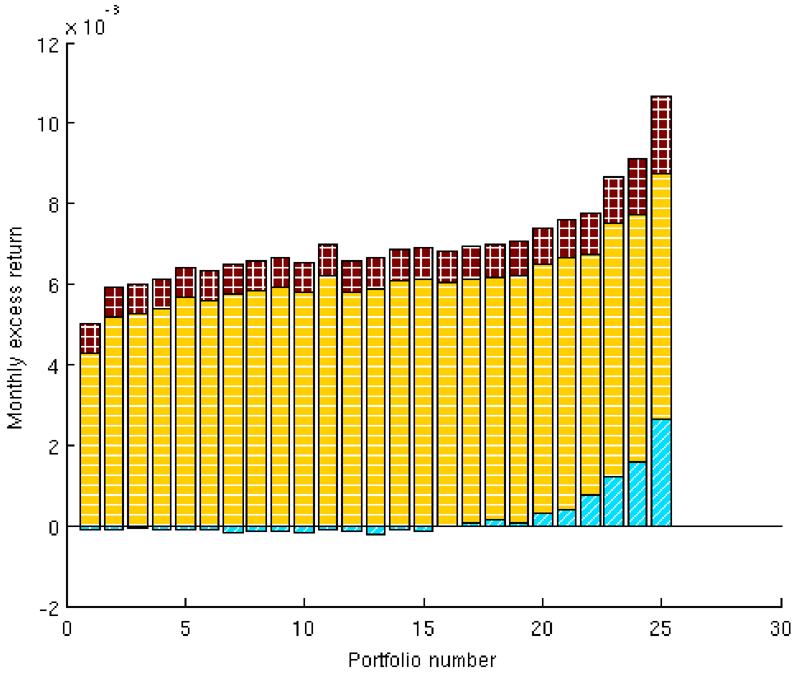

Figure 2: Decomposition of predicted excess return. The decomposition is given as the impact of the cost term (lower part), the covariance term (middle part), and the holding period variance term (upper part).

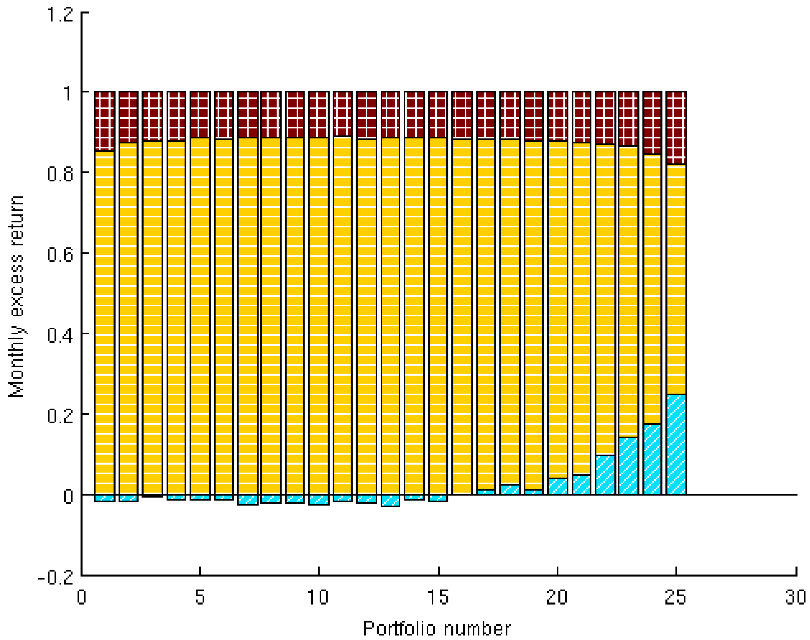

Figure 3: Decomposition of proportions of predicted excess return. The decomposition is given as the impact of the cost term (lower part), the covariance term (middle part), and the holding period variance term (upper part).